

# Hysteresis in a Superfluid Atom Circuit

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We study a hysteresis phenomenon in a rotating BEC with a weak link in a quasi-one-dimensional torus by proposing a microscopic theoretical model including a dissipation bath. By analyzing the role of dissipation and the decay rates of all the energy levels, we are able to give a microscopic interpretation of hysteresis recently observed in the experiment and confirm that the hysteresis is the result of the presence of metastable state. In particular, we obtain the hysteresis loops in a quench process just as that in the experiment. We also find that the shape and size of the hysteresis loop change drastically with the strength of the link.

## I. INTRODUCTION

Atomtronics [1–3] is an emerging interdisciplinary field that focuses on ultracold atom analogs of electronic circuits and devices. A series of theoretical demonstrations [4–7] and impressive experiments [8, 9] with BECs, a testing bed for atomtronics, have established this analogy, especially for Josephson effects [10–12], Bloch Oscillations [13]. Nevertheless, hysteresis in a superfluid atomtronic gas, which is considered to be essential to the realization of an atomic-gas superconducting quantum interference device (SQUID), has not been directly observed until recently [14]. In this experiment, both hysteresis and the quantization of flow have been observed in an atomtronic circuit formed from a ring of superfluid BEC obstructed by a rotating weak link. Just as the essential role it plays in the electronic circuit, the realization of hysteresis in atomtronic circuit will greatly accelerate the development of atomtronics because the controllability of hysteresis is crucial for the requirements of practical devices.

Hysteresis, widely used in electronic circuits, is the phenomenon where the state of a physical system depends upon its history. The canonical example of hysteresis in a classical system is that in ferromagnetism. According to the Landau theory of phase transitions [15]: the energy landscape, which changes with the applied magnetic field, is produced by calculating the energy of the system as a function of magnetization. Hysteresis occurs when the energy landscape has two local minima separated by an energy barrier. At some critical field, the barrier disappears and the system has only a global minimum.

At a rudimentary level, hysteresis manifests itself as the competition between experimental time scales and internal time scales, the former are determined by the frequencies of the applied perturbation and the latter are governed by relaxation, decay and so on [16]. To observe a hysteresis in a quantum system, we require that some eigenstates of the system are metastable, namely, we can

safely neglect the decays of those states in the experimental time scale. In this paper, we investigate the essential role that the metastable states play in the formation of hysteresis.

The swallowtail energy loop is a generic feature of hysteresis in an atomtronic circuit [17–20]. However, much of the study of swallowtails is rooted in the exact solutions to the Gross-Pitaevskii equation [21, 22]. Recently, the dynamics simulation in a toroidal BEC [23, 24] is also confined to the mean-field approximation.

In our study we introduce a microscopic model for the dissipation, and thus we can associate the swallowtail energy loop with the existence of metastable states. By this microscopic model, we obtain the relative decay rates of all many-body states, which decides whether a metastable state exists in our system or not, and leads to a quantitative calculation of the hysteresis in our system. Our calculations show that the interaction tends to increase the size of the hysteresis loop, while the strength of the link tends to decrease it, and confirm that there exists a metastable state which results in the hysteresis loops.

## II. THEORETICAL MODEL

### A. Two-mode approximation

We consider a quasi-one-dimensional dilute gas, containing  $N$  bosonic atoms in a thin annulus of radius  $R$  and cross-sectional radius  $r_0 \ll R$ , which rotates at frequency  $\Omega$  driven by a rotating repulsive potential  $V$  [25]. The Hamiltonian of this system  $\mathcal{S}$  in the rotating frame with frequency  $\Omega$  is given by

$$H_S = \sum_i \left[ \frac{L_{zi}^2}{2MR^2} + V(\vec{r}_i) \right] + \frac{1}{2} \sum_{ij} g \delta(\vec{r}_i - \vec{r}_j) - \Omega \sum_i L_{zi}, \quad (1)$$

where  $M$  is the atomic mass,  $g$  is the strength of contact interaction,  $L_z = -i\hbar\partial/\partial\theta$  is the angular momentum operator, and the potential takes the form

$$V = \begin{cases} V_0, & |\theta| \leq \theta_0, \\ 0, & |\theta| > \theta_0, \end{cases} \quad (2)$$

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which depletes the density in a small portion of the ring and thereby creates a weak link.

In terms of single-particle eigenstates  $\psi_l(\theta) = e^{il\theta}/\sqrt{2\pi}$  of  $L_z$  with angular momentum  $l\hbar$ , the Hamiltonian can be written as

$$H_S = \frac{1}{2}\hbar\Omega_0 \sum_j (j - \bar{\Omega})^2 a_j^\dagger a_j + \sum_{j \neq k} \frac{V_0 \sin(j-k)\theta_0}{(j-k)\pi} a_j^\dagger a_k + \frac{1}{2}g \sum_{jkm} a_j^\dagger a_k^\dagger a_{k-m} a_{j+m}, \quad (3)$$

where  $\Omega_0 = \hbar/MR^2$ ,  $\bar{\Omega} = \Omega/\Omega_0$ , and  $a_j^\dagger$  ( $a_j$ ) is the creation (annihilation) operator of boson with angular momentum  $j\hbar$ .

In the experiment [14], a two-step sequence is used to observe hysteresis in a BEC of  $^{23}\text{Na}$  atoms. The BEC is firstly prepared in either the  $n = 0$  or the  $n = 1$  circulation state by either not rotating the weak link or by rotating it at  $\Omega_1$ . Then the weak link is rotated at various angular velocities  $\Omega_2$  for a while.  $V_0$  is ramped to a certain value  $V_1$  in step 1 while to a chosen  $V_2$  in step 2. The transitions of average angular momentum  $\langle n \rangle = 0 \rightarrow 1$  and  $\langle n \rangle = 1 \rightarrow 0$  occur at different values of  $\Omega_2$  and form hysteresis loops.

To simulate the hysteresis in the experiment, we consider two quench processes of parameter  $\Omega$ :  $\Omega$  is ramped suddenly from 0 (or  $\Omega_1$ ) to a chosen  $\Omega_2$  and then the system is rotated at  $\Omega_2$  for a time period such that it arrives at its steady state. The quantity we describe the hysteresis is the average angular momentum

$$\hat{n} = \frac{L_z}{N} = \frac{\hat{n}_1}{N}. \quad (4)$$

To simplify the discussion, we consider just two single-particle eigenstates: the nonrotating state  $\psi_0(\theta)$  and the state  $\psi_1(\theta)$  with azimuthal angular momentum  $\hbar$ . In fact, since the energy of atoms is proportional to  $(j - \bar{\Omega})^2$ , as long as  $\bar{\Omega} < 1$ , atoms in either the  $\psi_0(\theta)$  or the  $\psi_1(\theta)$  state will have lower energies than those in the other states. According to the processes we consider, it is reasonable to assume that most atoms will stay in these two levels during the whole process. Then the Hilbert subspace is given by  $|n_0, n_1\rangle$  where  $n_0$  and  $n_1$ , denoting the numbers of bosons that occupy the states  $\psi_0(\theta)$  and  $\psi_1(\theta)$ , satisfy  $n_0 + n_1 = N$ . The Hamiltonian (3) can be approximated as

$$\bar{H}_S = \frac{1}{2}[\bar{\Omega}^2 \hat{n}_0 + (1 - \bar{\Omega})^2 \hat{n}_1] + \bar{u} \hat{A} + \bar{g} \hat{n}_0 \hat{n}_1, \quad (5)$$

where  $\hat{n}_0 = a_0^\dagger a_0$ ,  $\hat{n}_1 = a_1^\dagger a_1$ ,  $\hat{A} = a_0^\dagger a_1 + a_1^\dagger a_0$ ,  $\bar{u} = V_0 \sin \theta_0 / \pi \hbar \Omega_0$ ,  $\bar{g} = g / \hbar \Omega_0$ , and  $\bar{H}_S = H_S / \hbar \Omega_0$ .

## B. Model of dissipation

In the phenomenon of hysteresis, dissipation plays an essential role. Here we introduce a simple model to de-

scribe the dissipation of our system:

$$H_R = \sum_\mu \hbar \omega_\mu b_\mu^\dagger b_\mu, \quad (6)$$

$$H_I = \sum_\mu g_\mu \hat{A} (b_\mu + b_\mu^\dagger), \quad (7)$$

where  $H_R$  is the Hamiltonian of the reservoir  $\mathcal{R}$ , and  $H_I$  is the interaction between the reservoir  $\mathcal{R}$  and the system  $\mathcal{S}$ . It is obvious that the energy of  $\mathcal{S}$  can be dissipated into  $\mathcal{R}$  while the particle number of  $\mathcal{S}$  is conserved.

It is worthy to point out that the dissipation model given by Eq. (6) and Eq. (7) is a direct generalization of the Caldeira-Leggett model [26]. In fact, our dissipation model can be regarded as  $N$  bosonic two-level atoms interact with an ensemble of bosons, which becomes the standard Caldeira-Leggett model when  $N = 1$ . As argued by Caldeira and Leggett, their model is generally applicable when the coupling between the two-level system and the environment is sufficient weak, which is the main reason why we choose the dissipation model given in Eq. (6) and Eq. (7). In our case, for example, the main physical source of the environment may be a small number of atoms in the other modes except the two modes we consider.

Let  $|\Psi_a\rangle$ ,  $|\phi_\mu\rangle$  be the eigenstates of  $H_S$  and  $H_R$ , having eigenvalues  $E_a$  and  $E_\mu$  respectively. Let  $\rho(t)$  be the density operator at time  $t$  of the global system  $\mathcal{S} + \mathcal{R}$ , and  $\sigma(t)$  be the reduced density operator at time  $t$  of the system  $\mathcal{S}$ , which is defined by  $\sigma(t) = \text{Tr}_R \rho(t)$ . In the basis of eigenstates of  $H_S$ , the master equation is written as [27]:

$$\frac{d\sigma_{aa}}{dt} = \sum_{b \neq a} (\sigma_{bb} \Gamma_{b \rightarrow a} - \sigma_{aa} \Gamma_{a \rightarrow b}), \quad (8)$$

where  $\sigma_{aa}$  is the population of the energy level  $|\Psi_a\rangle$  of  $\mathcal{S}$ , and  $\Gamma_{a \rightarrow b}$  is the probability per unit time for the system  $\mathcal{S}$  to make a transition from level  $|\Psi_a\rangle$  to level  $|\Psi_b\rangle$  as a result of its coupling with  $\mathcal{R}$  for  $a \neq b$ , which is given by

$$\begin{aligned} \Gamma_{a \rightarrow b} &= \frac{2\pi}{\hbar} \sum_\nu |\langle \phi_\nu, \Psi_b | H_I | \text{Vac}, \Psi_a \rangle|^2 \delta(E_a - E_\nu - E_b) \\ &= \frac{2\pi}{\hbar} \sum_\nu g_\nu^2 |\langle \Psi_b | A | \Psi_a \rangle|^2 \delta(E_a - \hbar\omega_\nu - E_b) \\ &= \frac{2\pi}{\hbar} g^2(\omega_{ab}) \rho(\omega_{ab}) |\langle \Psi_b | A | \Psi_a \rangle|^2 \\ &= D |\langle \Psi_b | A | \Psi_a \rangle|^2, \end{aligned} \quad (9)$$

where the initial state of  $\mathcal{R}$  is in the vacuum state  $\rho_R(0) = |\text{Vac}\rangle\langle\text{Vac}|$ ,  $\omega_{ab} = \frac{E_a - E_b}{\hbar}$ , and  $\rho(\omega)$  is the energy spectrum at  $\omega$ . In the last line of Eq. (9), we assume that  $\rho(\omega)g^2(\omega)$  does not depend on  $\omega$ , and let  $D = \frac{2\pi}{\hbar} g^2(\omega_{ab}) \rho(\omega_{ab})$ .

According to Eq. (9), the conditions for the transition rate  $\Gamma_{a \rightarrow b} > 0$  are

$$E_a \geq E_b, \quad (10)$$

$$\langle \Psi_b | \hat{A} | \Psi_a \rangle \neq 0. \quad (11)$$

It means that the dissipation from  $|\Psi_a\rangle$  to  $|\Psi_b\rangle$  will occur if and only if the eigenenergy of  $|\Psi_a\rangle$  is above that of  $|\Psi_b\rangle$ , and the transition amplitude  $\langle\Psi_b|\hat{A}|\Psi_a\rangle$  is not zero.

If an eigenstate of  $\mathcal{S}$  that is not a ground state does not significantly dissipate to any state with lower energy in the time period we consider, then it is called a metastable state. We will show that the existence of metastable states is essential to the formation of hysteresis, which is discussed in Sec. III C. It needs to be stressed that the decay rate for the metastable state into the other lower levels may not equal to zero exactly, but it is far less than those of other excited levels. In the time period we consider, we can safely neglect the decay of the metastable state. Hence, the final state of our system is a probability distribution of the ground state and the metastable state, and the probability distribution depends on the history of our system, thus leading to the hysteresis, which is shown in Sec. III D.

### III. ANALYSIS OF HYSTERESIS

In this section, we first establish a general formalism for the analysis of hysteresis in our system. Then we use it to describe four different cases to explore the microscopic mechanism of hysteresis in our system by steps.

Since the hysteresis is the property of the steady state of a system, it is necessary to determine the steady state of our system by solving the master equation (8).

Let  $a \in \{0, 1, \dots, N\}$ , and assume that  $E_a > E_b$  if  $a > b$ . Then Eq. (8) can be rewritten as

$$\frac{d\sigma_{aa}}{dt} = -\sigma_{aa}\Gamma_a + \sum_{a < b} \sigma_{bb}\Gamma_{b \rightarrow a}, \quad (12)$$

where the total decay rate of the state  $|\Psi_a\rangle$  is

$$\Gamma_a = \sum_{b < a} \Gamma_{a \rightarrow b}. \quad (13)$$

If  $\forall a > 0$ ,  $\Gamma_a > 0$ , then the steady state will be the ground state  $|\Psi_0\rangle$ , and our system shows no hysteresis in the plane of  $\bar{\Omega}$ - $\langle\hat{n}\rangle$ .

According to Eq. (12), the population  $\sigma_{aa}$  in the steady state is not zero if and only if  $\Gamma_a = 0$ . Obviously, the decay rate of the ground state  $\Gamma_0 = 0$ . In addition, if the state  $|\Psi_c\rangle$  is a metastable state in our system, then also we have  $\Gamma_c = 0$ . In the steady state derived from Eq. (12), the population of a state without decay is

$$\sigma_{cc}(\infty) = \sigma_{cc}(0) + \sum_{a > c} \sigma_{aa}(0)p_{a \rightarrow c}, \quad (14)$$

where the probability of decay from  $|\Psi_a\rangle$  to  $|\Psi_c\rangle$  is

$$p_{a \rightarrow c} = \sum_{a > b_1 > b_2 > \dots > b_k > c} \gamma_{a \rightarrow b_1} \gamma_{b_1 \rightarrow b_2} \dots \gamma_{b_k \rightarrow c} \quad (15)$$

with  $\gamma_{a \rightarrow b} = \frac{\Gamma_{a \rightarrow b}}{\Gamma_a}$ . In other words,  $p_{a \rightarrow c}$  is the decay probability from  $|\Psi_a\rangle$  to  $|\Psi_c\rangle$  over all possible decay paths.

The average angular momentum in the steady state is

$$\langle\hat{n}\rangle = \sum_{c: \Gamma_c=0} \langle\Psi_c|\hat{n}|\Psi_c\rangle \left( \sigma_{cc}(0) + \sum_{a > c} \sigma_{aa}(0)p_{a \rightarrow c} \right). \quad (16)$$

In particular, when there does not exist metastable states in our system, Eq. (16) becomes

$$\langle\hat{n}\rangle = \langle\Psi_0|\hat{n}|\Psi_0\rangle, \quad (17)$$

which implies that  $\langle\hat{n}\rangle$  does not depend on the initial state, and there can not exist any hysteresis in the  $\bar{\Omega}$ - $\langle\hat{n}\rangle$  plane. Otherwise, Eq. (16) implies that  $\langle\hat{n}\rangle$  depends on the initial condition, and it usually forms a hysteresis in the  $\bar{\Omega}$ - $\langle\hat{n}\rangle$  plane.

#### A. The case of $\bar{g} = \bar{u} = 0$

In this case, the eigenstates of  $\bar{H}_S$  are

$$|\Psi_{n_0}\rangle = |n_0, n_1\rangle \quad (18)$$

for  $n_0 = 0, 1, \dots, N$  with eigenenergies

$$E_{n_0} = \frac{1}{2}[\bar{\Omega}^2 n_0 + (1 - \bar{\Omega})^2 n_1]. \quad (19)$$

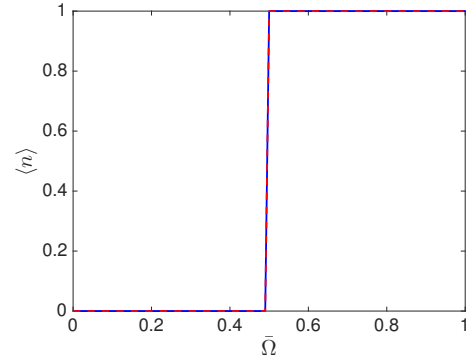


FIG. 1: (Color online) The average momentum  $\langle\hat{n}\rangle$  via  $\bar{\Omega}$  with parameters  $N = 1000$ ,  $\bar{u} = 0$ , and  $\bar{g} = 0$ . The solid blue line and the dashed red line represents different initial states  $|N, 0\rangle$  and  $|0, N\rangle$  respectively.

When  $\bar{\Omega} < \frac{1}{2}$ ,  $E_{n_0} < E_{n_0-1}$ , and  $|N, 0\rangle$  is the ground state. Because  $\forall n_0 \neq N$ , the amplitude  $\langle\Psi_{n_0+1}|A|\Psi_{n_0}\rangle \neq 0$ , which implies that  $|\Psi_{n_0}\rangle$  is not a metastable state. Hence the steady state at  $\bar{\Omega} (< \frac{1}{2})$  is  $|N, 0\rangle$ , and then the average angular momentum  $\langle\hat{n}\rangle = 0$ .

When  $\bar{\Omega} > \frac{1}{2}$ ,  $E_{n_0} > E_{n_0-1}$ , and  $|0, N\rangle$  is the ground state. Because  $\forall n_0 \neq 0$ , the amplitude  $\langle\Psi_{n_0-1}|A|\Psi_{n_0}\rangle \neq 0$ , which implies that  $|\Psi_{n_0}\rangle$  is not a metastable state.

Hence the steady state at  $\bar{\Omega} (> \frac{1}{2})$  is  $|0, N\rangle$ , and then the average angular momentum  $\langle \hat{n} \rangle = 1$ . Because at any  $\bar{\Omega}$ , the steady state does not depend on its history, and there will not exist hysteresis, but the average momentum will make a transition between 0 and 1 at  $\bar{\Omega} = \frac{1}{2}$ , see a demonstration with  $N = 1000$  in Fig. 1.

### B. The case of $\bar{g} = 0, \bar{u} \neq 0$

In this case, the Hamiltonian  $\bar{H}_S$  can be diagonalized as

$$\begin{aligned} \bar{H}_S &= \frac{1}{2}[\bar{\Omega}^2 a_0^\dagger a_0 + (1 - \bar{\Omega})^2 a_1^\dagger a_1] + \bar{u}(a_0^\dagger a_1 + a_1^\dagger a_0) \\ &= \begin{pmatrix} a_0^\dagger & a_1^\dagger \end{pmatrix} \begin{pmatrix} \frac{1}{2}\bar{\Omega}^2 & \bar{u} \\ \bar{u} & \frac{1}{2}(1 - \bar{\Omega})^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ &= \epsilon_+ c_+^\dagger c_+ + \epsilon_- c_-^\dagger c_-, \end{aligned} \quad (20)$$

where

$$\epsilon_\pm = \frac{1}{4} \left[ \bar{\Omega}^2 + (1 - \bar{\Omega})^2 \pm \sqrt{16\bar{u}^2 + (1 - 2\bar{\Omega})^2} \right], \quad (21)$$

and

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \quad (22)$$

with  $\theta = \arctan \frac{4\bar{u}}{2\bar{\Omega}-1}$ .

The eigenvectors and eigenvalues of  $\bar{H}_S$  are

$$|\Psi_{n_+}\rangle = |n_+, n_-\rangle, \quad (23)$$

$$E_{n_+} = \epsilon_+ n_+ + \epsilon_- n_-. \quad (24)$$

Because  $\epsilon_+ > \epsilon_-$ , we have  $E_{n_+} > E_{n_+-1}$ , and  $|\Psi_0\rangle$  is the ground state.

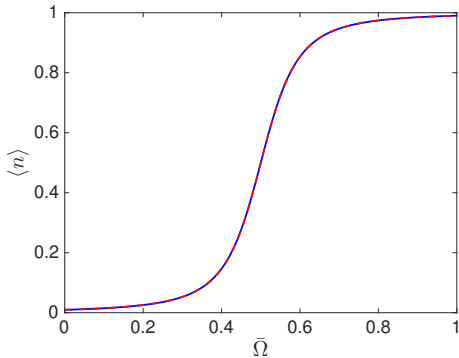


FIG. 2: (Color online) The average momentum  $\langle n \rangle$  via  $\bar{\Omega}$  with parameters  $N = 1000$ ,  $\bar{u} = 0.05$ , and  $\bar{g} = 0$ .

According to Eq. (22), we obtain  $\forall n_+ > 0$ ,

$$\langle \Psi_{n_+-1} | A | \Psi_{n_+} \rangle = \sqrt{n_+(N - n_+ + 1)} \cos \theta > 0, \quad (25)$$

which implies that  $|\Psi_{n_+}\rangle$  is not metastable, i.e., there does not exist metastable states in our system. Thus

the steady state is the ground state  $|\Psi_0\rangle$ , which does not depend on the initial conditions. Then the average angular momentum in the ground state is

$$\langle \hat{n} \rangle = \frac{1 + \cos \theta}{2} = \frac{1}{2} + \frac{\bar{\Omega} - 1/2}{\sqrt{16\bar{u}^2 + (2\bar{\Omega} - 1)^2}}, \quad (26)$$

which is numerically demonstrated in Fig. 2 with parameters  $N = 1000$  and  $\bar{u} = 0.05$ .

### C. The case of $\bar{g} \neq 0, \bar{u} = 0$

In this case, the eigenstates and eigenenergies of  $\bar{H}_S$  are

$$|\Psi_{n_0}\rangle = |n_0, n_1\rangle, \quad (27)$$

$$E_{n_0} = \frac{1}{2}[\bar{\Omega}^2 n_0 + (1 - \bar{\Omega})^2 n_1] + \bar{g} n_0 n_1. \quad (28)$$

Let us analyze the condition for the appearance of a metastable state in our system. Because

$$\begin{aligned} \langle \Psi_m | \hat{A} | \Psi_{n_0} \rangle &= \delta_{m, n_0+1} \sqrt{(n_0 + 1)(N - n_0)} \\ &\quad + \delta_{m, n_0-1} \sqrt{n_0(N - n_0 + 1)}, \end{aligned}$$

the state  $|\Psi_{n_0}\rangle$  is not a metastable state if and only if  $E_{n_0+1} < E_{n_0}$  or  $E_{n_0-1} < E_{n_0}$ . In other words,  $|\Psi_{n_0}\rangle$  is a metastable state if and only if  $n_0$  is a local minimum of the function  $E_{n_0}$ . Since  $\frac{d^2 E}{dn_0^2} = -\bar{g} < 0$ ,  $E_{n_0}$  will take a local maximum when  $\frac{dE}{dn_0} = 0$ . Hence only the two ends of  $n_0$ , i.e.  $n_0 = 0$  and/or  $n_0 = N$ , may be a local minimum of  $E_{n_0}$ . Therefore the condition for the existence of a metastable state is  $E_0 < E_1$  and  $E_N < E_{N-1}$ , which implies that

$$\Omega_c^- < \bar{\Omega} < \Omega_c^+, \quad (29)$$

with

$$\Omega_c^\pm = \frac{1}{2} \pm \bar{g}(N - 1). \quad (30)$$

When  $\Omega_c^- < \bar{\Omega} < \frac{1}{2}$ ,  $E_N < E_0$ , and then  $|N, 0\rangle$  is the ground state and  $|0, N\rangle$  is the metastable state. When  $\frac{1}{2} < \bar{\Omega} < \Omega_c^+$ ,  $E_0 < E_N$ , and then  $|0, N\rangle$  is the ground state and  $|N, 0\rangle$  is the metastable state.

When our system is initially prepared in the ground state  $|N, 0\rangle$  by choosing  $\bar{\Omega} = 0$ , the average angular momentum  $\langle \hat{n} \rangle = 0$ . As  $\bar{\Omega}$  is quenched to the region  $(0, \frac{1}{2})$ ,  $|N, 0\rangle$  is still the ground state with  $\langle \hat{n} \rangle = 0$ . As  $\bar{\Omega}$  is quenched to the region  $(\frac{1}{2}, \Omega_c^+)$ ,  $|N, 0\rangle$  is not a ground state but it is metastable, and we still have  $\langle \hat{n} \rangle = 0$ . As  $\bar{\Omega}$  is quenched to the region  $(\Omega_c^+, 1)$ , then  $|N, 0\rangle$  is not stable or metastable, and the system evolves into the ground state  $|0, N\rangle$  with  $\langle \hat{n} \rangle = 1$ . Similarly, we can analyze the case when  $\bar{\Omega}$  is quenched back from 1 to 0, and we find that  $\langle \hat{n} \rangle = 1$  when  $\bar{\Omega} \in (\Omega_c^-, 1)$ , and  $\langle \hat{n} \rangle = 0$  when  $\bar{\Omega} \in (0, \Omega_c^-)$ . Therefore it forms the hysteresis in the  $\bar{\Omega} - \langle \hat{n} \rangle$  plane, see Fig. 3 for a demonstration with  $N = 1000$  and  $\bar{g} = 0.0001$ .

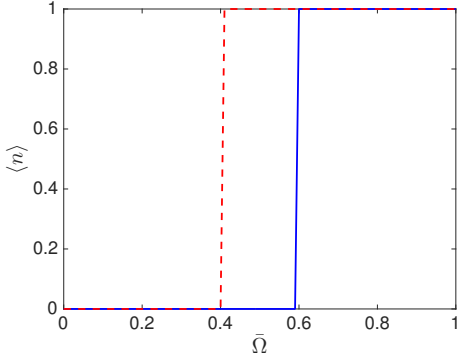


FIG. 3: (Color online) The average momentum  $\langle n \rangle$  via  $\bar{\Omega}$  with parameters  $N = 1000$ ,  $\bar{u} = 0$ , and  $\bar{g} = 0.0001$ .

#### D. The case of $\bar{g} \neq 0, \bar{u} \neq 0$

In this general case, we can not solve the eigen problem of  $\bar{H}_S$  analytically, and must resort to the numerical method.

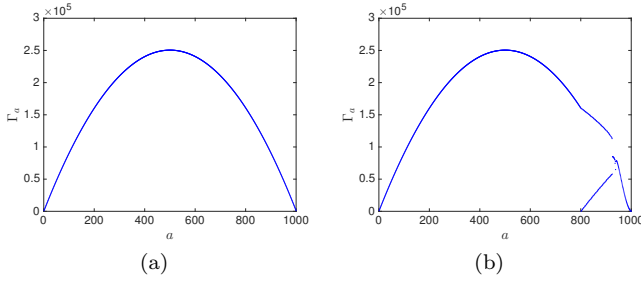


FIG. 4: (Color online) The decay rates of excited states with parameters  $N = 1000$ ,  $\bar{g} = 0.0001$ ,  $\bar{u} = 0.0002$ , and (a)  $\bar{\Omega} = 0.35$ ; (b)  $\bar{\Omega} = 0.42$ .

First, we need to calculate all the eigenenergies and eigenstates of  $\bar{H}_s$ , and determine whether there is a metastable state in our system, which can be obtained from the decay rates of excited states of  $\bar{H}_S$ . If there is a metastable state, then there is an excited state with near zero decay rate. Here we give an example on how to numerically confirm the existence of metastable states. When  $N = 1000$ ,  $\bar{g} = 0.0001$ , and  $\bar{u} = 0.0002$ , the decay rates of excited states with  $\bar{\Omega} = 0.35$  and  $\bar{\Omega} = 0.42$  are shown in Fig. 4a and Fig. 4b respectively. In Fig. 4a the two smallest decay rates are  $\Gamma_1 = 999.997949$  and  $\Gamma_{1000} = 999.872868$ , which implies that there does not exist metastable states in this case. In Fig. 4b, the four smallest decay rates are given by  $\Gamma_1 = 999.96427$ ,  $\Gamma_{803} = 1000.597295$ ,  $\Gamma_{1000} = 157.236302$ , and  $\Gamma_{801} \simeq 0$ , which implies that the 801-th eigenstate is metastable.

Then we calculate the average momentum  $\langle n \rangle$  as a function of  $\bar{\Omega}$  to show the hysteresis in our system. As shown in Sec. III C, the hysteresis will appear when  $\bar{g} \neq 0$  and  $\bar{u} = 0$  demonstrated in Fig. 3. Now we examine how

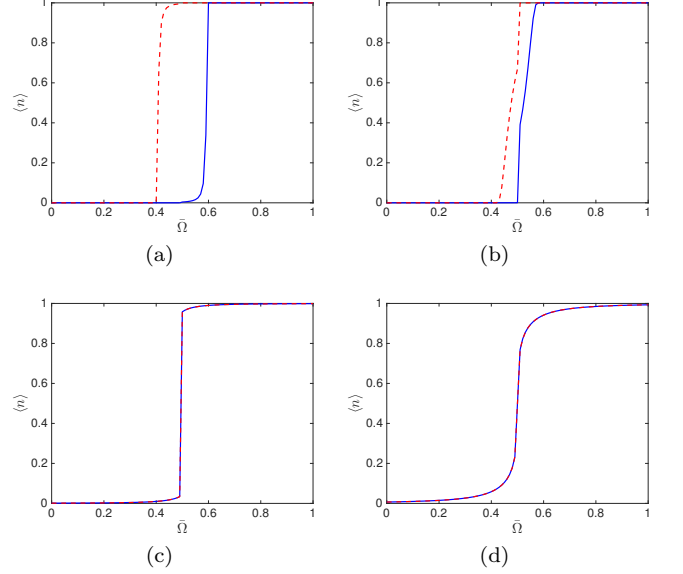


FIG. 5: (Color online) The average momentum  $\langle n \rangle$  via  $\bar{\Omega}$  with parameters  $N = 1000$ ,  $\bar{g} = 0.0001$ , and (a)  $\bar{u} = 0.0002$ ; (b)  $\bar{u} = 0.002$ ; (c)  $\bar{u} = 0.02$ ; (d)  $\bar{u} = 0.05$ .

the hysteresis changes with the increasing of  $\bar{u}$  for given parameters  $N$  and  $\bar{g}$ , which is demonstrated in Fig. 5 with different  $\bar{u}$ . In the case of  $N = 1000$  and  $\bar{g} = 0.0001$ , there is a regular hysteresis in the plane of  $\bar{\Omega} - \langle n \rangle$  when  $\bar{u} = 0.0002$  shown in Fig. 5a, which is smoother than that in Fig. 3. When  $\bar{u} = 0.002$ , the hysteresis becomes narrower demonstrated in Fig. 5b. When  $\bar{u} = 0.02$  in Fig. 5c, the hysteresis almost disappears. When  $\bar{\Omega} = 0.05$  shown in Fig. 5d, the behavior of  $\langle n \rangle$  becomes similar as that in Fig. 2, and the hysteresis completely disappears.

As shown above, the appearance of metastable state is essential to the formation of hysteresis. Thus it is natural to characterize the degree of hysteresis in our system with the width of the region with metastable states. Denote the region of  $\bar{\Omega}$  with metastable states with  $[\Omega_c^-, \Omega_c^+]$ , and the degree of hysteresis becomes

$$\Delta\bar{\Omega} = \Omega_c^+ - \Omega_c^-. \quad (31)$$

Based on the concept of the degree of hysteresis, we investigate how  $\bar{u}$  changes the degree of hysteresis for given  $N$  and  $\bar{g}$ , which is given in Fig. 6. In particular, it defines a critical point  $\bar{u}_c$  such that  $\bar{u} < \bar{u}_c$  is a condition for the appearance of hysteresis. For example, when  $N = 1000$  and  $\bar{g} = 0.0001$ , the critical point  $\bar{u}_c = 0.046$ .

Furthermore, we present the degree of hysteresis  $\Delta\bar{\Omega}$  for different parameters when  $N = 1000$ . Fig. 7 suggests that  $\Delta\bar{\Omega}$  is determined by the competition between  $\bar{g}$  and  $\bar{u}$ : the former tends to increase  $\Delta\bar{\Omega}$  while the latter decrease it. Also, we notice that  $\bar{u}_c(\bar{g}) \propto \bar{g}$  (the boundary between the parameter region where  $\Delta\bar{\Omega} > 0$  and the blank space in Fig. 7). In fact, for  $\forall \bar{g}$ , if  $\bar{u}$  is set to be  $\bar{u}_c(\bar{g})$ ,  $\Omega_c^+ = \Omega_c^- = 1/2$ , it means that  $\bar{u}_c(\bar{g})$  is determined

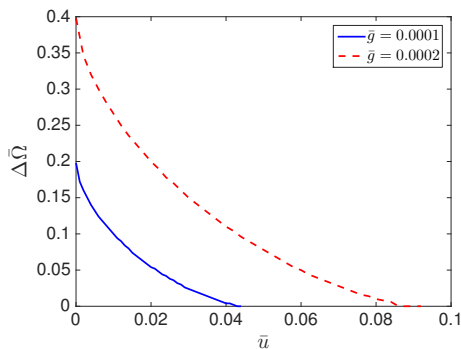


FIG. 6: (Color online) The degree of hysteresis via  $\bar{u}$  with parameter  $N = 1000$ . The cases of  $\bar{g} = 0.0001$  and  $\bar{g} = 0.0002$  are denoted by the blue solid line and the red dashed line respectively.

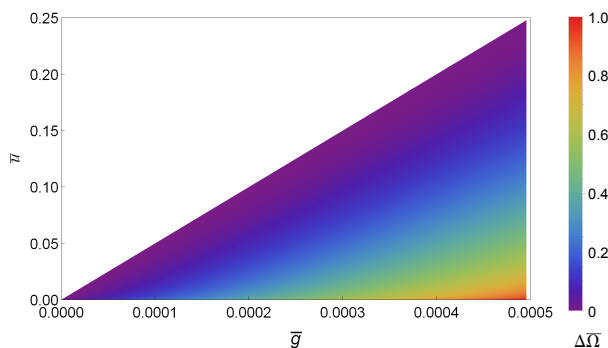


FIG. 7: (Color online) The parameter regions where  $\Delta\bar{\Omega} > 0$  and the degree of hysteresis for different parameters with  $N = 1000$ .

by the energy band structure of Hamiltonian (5) for  $\bar{\Omega} = 1/2$ , i.e.,  $\bar{H} = N/8 + \bar{g}a_0^\dagger a_0 a_1^\dagger a_1 + \bar{u}(a_0^\dagger a_1 + a_1^\dagger a_0)$ , it's obvious that the energy band structure is determined by  $\bar{u}/\bar{g}$ , leading to  $\bar{u}_c(\bar{g}) \propto \bar{g}$ .

#### IV. DISCUSSION AND SUMMARY

In our present treatment of hysteresis, we focuses on the instantaneous change of  $\bar{\Omega}$ , which simplifies our calculation of hysteresis and simulates well with the present

experiment. However, it is worthy to point out that this is not the unique process to observe the hysteresis in our system. For example, another interesting process for this purpose is the adiabatic change of  $\bar{\Omega}$  with a slow variant rate. In this adiabatic process, the many-body Landau-Zener tunneling [28–30] may be involved to affect the hysteresis, which needs to be investigated further in future.

In summary, we have presented a microscopic theory of the hysteresis in an atomic BEC, in which a two-mode model is used to describe the BEC system in a ring with an external potential, and an ensemble of bosons is introduced to act as the environment of the BEC system. We find that such a simple two-mode model captures the essence of hysteresis, an effect from the interactions between atoms and the dissipation induced by its environment. In particular, the existence of an metastable state is essential for the formation of hysteresis in our system. More precisely, the steady state depends on the initial state when a metastable state appears, which leads to the hysteresis in the average momentum  $\langle \hat{n} \rangle$  as a function of the rotation frequency  $\bar{\Omega}$ . We also find that the parameter of the external potential  $\bar{u}$ , which controls the tunneling between the two modes we consider, weakens the effect of hysteresis in our system. In particular, for a given particle number  $N$  and the interaction constant  $\bar{g}$ , there is a critical potential  $\bar{u}_c$  such that if the potential is beyond the critical value, the hysteresis will be completely destroyed. We numerically give the critical potential parameter  $\bar{u}_c$  for different interaction parameter  $\bar{g}$  when  $N = 1000$ , and observe that  $\bar{u}_c \propto \bar{g}$ , which is also proved theoretically. It is worthy to point out that the predictions of our microscopic theory are consistent with the observations in the present experiment. We hope that our theory of hysteresis will promote our understandings of hysteresis at the microscopic level, and gives more rich and precise predictions beyond the mean field theory.

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